Buckling of a Long Cylindrical Shell Containing an Elastic Core

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A long, thin, circular cylindrical shell containing an elastic core, bonded to the inner surface of the shell, is subjected to a uniform external hydrostatic pressure. Stability of equilibrium of the shell is investigated by considering possible neighboring equilibrium states. The loading exerted by the elastic core on the shell in the deformed state is found by solving an associated boundary value problem of the linearized theory of elasticity in the presence of initial stresses. An expression for the buckling pressure of the shell is derived, and results are presented for a wide range of shell and medium parameters.

Nomenclature

a = shell radius
$D = Eh^3/12(1 - \nu^2)$ = shell flexural modulus
E, E_c = Young's modulus for shell and core, respectively
E_j = unit elongation of line element originally in j direction
$E_p = Eh/(1 - \nu^2)$ = shell compressional modulus
f_θ, f_r = tangential and radial components of boundary traction taken in undeformed position per unit original area
$\Delta F_\theta, \Delta F_r$ = circumferential and radial components of change caused by deformation of shell surface tractions per unit original area
$\Delta F_\theta^i, \Delta F_r^i$ = circumferential and radial components of change caused by deformation of initial uniform pressure on shell surfaces
F_θ^a, F_r^a = additional shear and normal components of stress at shell-core interface induced by shell displacements
G, H, L, M, R, S = dimensionless parameters defined by Eq. (13)
h = shell thickness
$\mathbf{k}_r, \mathbf{k}_\theta$ = unit vectors tangent to undeformed coordinate lines
$\mathbf{k}_r', \mathbf{k}_\theta'$ = unit vectors tangent to deformed coordinate lines
$\Delta m_\theta = -h/2\Delta F_\theta$ = increment of moment corresponding to ΔF_θ, measured positive clockwise
n = buckling mode number
N = circumferential shell stress resultant
p = critical pressure of long, thin, circular cylindrical shell containing solid, elastic core
p' = pressure exerted by core on shell in its undeformed position
p_0 = critical pressure of long, thin, circular cylindrical shell subjected to uniform hydrostatic pressure
r, θ = cylindrical coordinates
u_θ, u_r = tangential and radial displacements in elastic core
v, w = tangential and radial displacement components of middle surface of shell measured positive clockwise and radially outward, respectively
α = dimensionless parameter defined by Eq. (18)
Δ = dilatation in linear theory of elasticity
δ_{ij} = Kronecker's delta
λ, μ = Lamé elastic constants
ν, ν_c = Poisson's ratio for shell and core, respectively
σ_{ij} = Trefftz components of stress per unit original area
σ_{ij}^a = additional stresses in elastic core induced by shell displacements
τ_{ij} = components of stress per unit original area shown in Fig. 2
ω = rotation in linear theory of elasticity

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1. Introduction

A LONG, thin, circular cylindrical shell containing a solid, elastic core, Fig. 1, bonded to the inner surface of the shell, is subjected to a uniform hydrostatic pressure. Stability of equilibrium of the shell is investigated by examining possible adjacent equilibrium states. The buckling load is then defined as the smallest load that admits a nonaxially symmetric neighboring equilibrium configuration for a cylindrical shell, which initially has a perfectly circular cross section. The initial equilibrium state for the shell is defined by the middle surface coordinate $r = a$, the applied hydrostatic pressure p on the outer shell surface, and the uniform reaction from the core p'. Since the shell displacements associated with the initial surface pressures are neglected in most applications (e.g., see Ref. 1), the position of initial equilibrium will be called the undeformed position. The adjacent equilibrium state will be called the position of final equilibrium or the deformed position.

For a solid core, the contained elastic medium is initially in equilibrium under an isotropic, homogeneous state of stress given by the pressure p'. (Only solid cores are considered in this paper, but the methods presented may be extended to hollow cores.) If, however, the applied hydrostatic pressure p is the critical pressure, an adjacent equilibrium position of the shell exists and is defined by the tangential and radial middle surface shell displacements. Because the shell and core are in contact, the shell displacements induce additional surface tractions and produce changes in the initial surface pressure at the shell-core interface. The resulting distributed force system on the inner shell surface in its deformed position is determined from the linearized theory for an elastic body under initial stress. These surface tractions are then related to the shell displacements, and the shell equations presented by Armenákas and Herrmann[2] are used to derive an expression for the critical hydrostatic pressure.

This expression contains the elastic material constants of the shell and core, the radius-to-thickness ratio of the shell, and the buckling mode number.

The buckling of a long cylindrical shell inclosing an elastic medium was investigated previously by Seide and Weingarten.[3] That analysis used the linear theory of elasticity to determine the reaction from the core and neglected shear stresses at the shell-core interface. Since the linear theory neglects changes in the geometry of a deformed elastic element, the initial surface tractions from the core act as a constant-directional pressure (e.g., see Ref. 4). However, the present analysis shows that, for high values of Poisson's ratio of the core, these initial surface tractions act as a hydrostatic pressure.

2. Shell Equations

Several linearized theories of motion for elastic, circular cylindrical shells subjected to a general state of initial stress have been advanced by Herrmann and Armenákas.[5] In Ref. 2, these equations were applied to study the effect of several particular states of initial stress on the dynamic response of an infinitely long shell, i.e., the motion was independent of the axial coordinate. For a thin shell of radius a and thickness h, subjected to an initial state of uniform lateral pressure, the equations in Ref. 2 for equilibrium reduce to

$$(E_p + N)\frac{\partial^2 v}{\partial \theta^2} - Nv + (E_p + 2N)\frac{\partial w}{\partial \theta} + a^2 \Delta F_\theta = 0 \quad (1a)$$

$$(E_p + 2N)\frac{\partial v}{\partial \theta} + E_p w + N\left(w - \frac{\partial^2 w}{\partial \theta^2}\right) + \frac{D}{a^2}\left(w + 2\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^4 w}{\partial \theta^4}\right) - a^2 \Delta F_r - a\frac{\partial}{\partial \theta}(\Delta m_\theta) = 0 \quad (1b)$$

where E_p is the shell compressional modulus, D is the shell flexural modulus, N is the initial uniform circumferential stress resultant, and v, w are the circumferential and radial middle surface displacements, measured positive clockwise and radially outward, respectively. The terms ΔF_θ, ΔF_r are the circumferential and radial components of the change caused by deformation of the shell surface traction, taken per unit undeformed middle surface area, and Δm_θ is the increment of moment corresponding to ΔF_θ, measured positive clockwise. In the notation of Ref. 5,

$$\Delta F_\theta = \Delta F_\theta{}^i + F_\theta{}^a \qquad \Delta F_r = \Delta F_r{}^i + F_r{}^a$$

$$\Delta m_\theta = -h/2\, \Delta F_\theta$$

The quantities $\Delta F_\theta{}^i$, $\Delta F_r{}^i$ consist of the changes related to the initial uniform pressure on the outer shell surface and the initial surface tractions from the core, whereas $F_\theta{}^a$, $F_r{}^a$ are the additional shear and normal components of stress at the shell-core interface directly induced by the shell displacements.

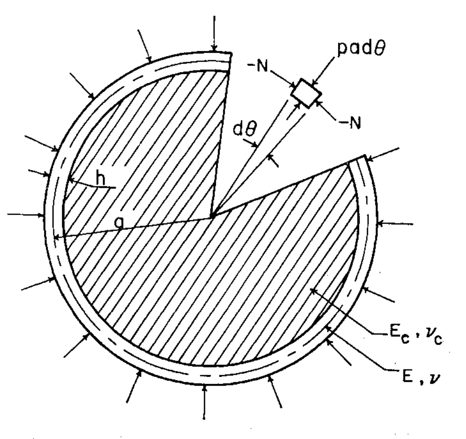


Fig. 1. Cross section of shell containing an elastic core.

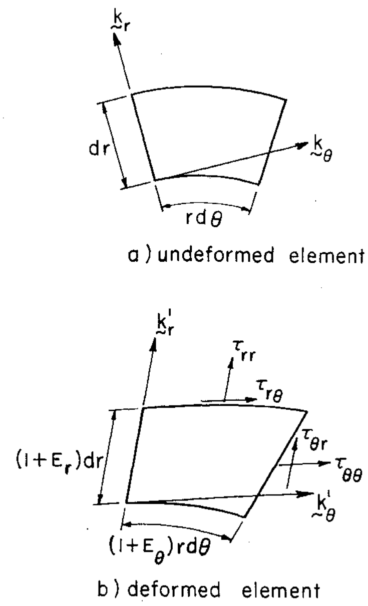


Fig. 2. Components of stress per unit original area.

For the solution presented in Ref. 3, the changes caused by deformation of the initial surface tractions exerted by the elastic core on the shell were neglected. In the present analysis, the values of $\Delta F_\theta{}^i$, $\Delta F_r{}^i$ are unknown a priori and are determined by solving an appropriate boundary value problem for the elastic core. Thus, ΔF_θ, ΔF_r will depend on the conditions at the shell-medium interface and the elastic constants of the core.

3. Equations of Elastic Core

The linearized equations for an elastic medium in the presence of initial stress are developed from the nonlinear theory of elasticity. Equilibrium equations and boundary conditions for the nonlinear theory, referred to a cylindrical coordinate system, can be obtained from the principle of virtual displacements, as was done by Novozhilov[6] with reference to a Cartesian system. For a state of plane strain the stress equilibrium equations are

$$\sigma_{rr}\left(1 + \frac{\partial u_r}{\partial r} + r\frac{\partial^2 u_r}{\partial r^2}\right) + \frac{\partial \sigma_{rr}}{\partial r}\left(r + r\frac{\partial u_r}{\partial r}\right) - \sigma_{\theta\theta}\left(1 + \frac{u_r}{r} + \frac{2}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{1}{r}\frac{\partial^2 u_r}{\partial \theta^2}\right) + \frac{\partial \sigma_{\theta\theta}}{\partial \sigma \theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) + 2\sigma_{r\theta}\left(\frac{\partial^2 u_r}{\partial r \partial \theta} - \frac{\partial u_\theta}{\partial r}\right) + \frac{\partial \sigma_{r\theta}}{\partial r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial \sigma_{r\theta}}{\partial \theta}\left(1 + \frac{\partial u_r}{\partial r}\right) = 0 \quad (2a)$$

$$\sigma_{rr}\left(\frac{\partial u_\theta}{\partial r} + r\frac{\partial^2 u_\theta}{\partial r^2}\right) + \frac{\partial \sigma_{rr}}{\partial r} r \frac{\partial u_r}{\partial r} + \sigma_{\theta\theta}\left(\frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\left(1 + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + 2\sigma_{r\theta}\left(1 + \frac{\partial u_r}{\partial r} + \frac{\partial^2 u_\theta}{\partial r \partial \theta}\right) + \frac{\partial \sigma_{r\theta}}{\partial \theta}\frac{\partial u_\theta}{\partial r} + \frac{\partial \sigma_{r\theta}}{\partial r}\left(r + u_r + \frac{\partial u_\theta}{\partial \theta}\right) = 0 \quad (2b)$$

where u_θ, u_r are the displacement components, and σ_{ij} are the Trefftz components of stress which form a symmetric tensor. The physical significance of the Trefftz components of stress becomes apparent by considering a deformed elastic element, such as that shown in Fig. 2b. If the force vector acting on the surface of a deformed element is resolved into nonorthogonal components parallel to the unit vectors $\mathbf{k}_r'$, $\mathbf{k}_\theta'$, the stress components τ_{ij} are defined as these force components divided by the face area before deformation. The Trefftz components of stress are then related to the matrix τ_{ij} by

$$\sigma_{ij} = \tau_{ij}/(1 + E_j)$$

where E_j is the unit elongation of a line element originally in the j direction.

The components of traction on the surface $r = a, f_r, f_\theta$, taken parallel to the unit vectors $\mathbf{k}_r$, $\mathbf{k}_\theta$ (Fig. 2a) and per unit undeformed surface area, are related to the Trefftz components of stress by

$$f_r = \left(1 + \frac{\partial u_r}{\partial r}\right)\sigma_{rr} + \frac{1}{a}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\sigma_{r\theta} \quad \text{at } r = a \quad (3a)$$

$$f_\theta = \left(1 + \frac{u_r}{a} + \frac{1}{a}\frac{\partial u_\theta}{\partial \theta}\right)\sigma_{r\theta} + \frac{\partial u_\theta}{\partial r}\sigma_{rr} \quad \text{at } r = a \quad (3b)$$

Equations (2) and (3) now can be used to establish a set of linearized equations for an elastic body under high initial stresses subject to small additional disturbances. It is first assumed that the general deformed configuration is reached from an unstressed and unstrained state by passing through an intermediate equilibrium state, the state of initial stress. For a solid core, the elastic medium is in initial equilibrium under an isotropic, homogeneous state of stress given by the pressure p'. The deformations associated with the initial pressure p' then are neglected, and the medium is allowed to reach its final equilibrium position by assuming small deviations from the position of initial equilibrium. This final equilibrium state is defined by the additional displacements u_r, u_θ, which produce small strains and rotations, and the stresses $\sigma_{ij} = -p'\delta_{ij} + \sigma_{ij}^a$. The linearized equilibrium equations are obtained by substituting $\sigma_{ij} = -p'\delta_{ij} + \sigma_{ij}^a$ into Eqs. (2) and neglecting all of the products of the additional stresses σ_{ij}^a and displacement gradients but retaining all of the products of the initial pressure p' and displacement gradients. Thus, the linearized equilibrium equations are

$$\frac{\partial \sigma_{rr}^a}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}^a}{\partial \theta} + \frac{\sigma_{rr}^a - \sigma_{\theta\theta}^a}{r} - p'\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}\right) = 0 \quad (4a)$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}^a}{\partial \theta} + \frac{2}{r}\sigma_{r\theta}^a + \frac{\partial \sigma_{r\theta}^a}{\partial r} - p'\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}\right) = 0 \quad (4b)$$

Since the additional strains and rotations are small as compared to unity, it is assumed that the differences between the initial and final stresses are related to the additional strains in accordance with Hooke's law for an isotropic, elastic solid. Guided by the work by Biot,[7] the differences between the initial pressure and the Trefftz components of stress σ_{ij} are taken to be proportional to the additional strains, i.e., σ_{ij}^a are taken proportional to the additional strains. Then, using the stress-displacement relations for small strains and rotations (e.g., see Ref. 8), the linearized equilibrium equations in terms of displacements may be written in the form

$$(\lambda + 2\mu - p')(\partial\Delta/\partial r) - 2(\mu - p')(1/r)(\partial\omega/\partial\theta) = 0 \quad (5a)$$

$$(\lambda + 2\mu - p')(1/r)(\partial\Delta/\partial\theta) + 2(\mu - p')(\partial\omega/\partial r) = 0 \quad (5b)$$

where λ, μ are the Lamé constants, Δ is the dilatation, and ω is the rotation. Thus, the initial pressure p' merely has the effect of modifying the elastic constants; this also is pointed out in Ref. 9. Furthermore, since p' is usually much less than the values of λ, μ, the equilibrium equations governing the classical linear theory of elasticity can be employed to calculate the additional stresses and displacements.

The linearized tractions on the surface $r = a$ are

$$f_r = -[1 + (\partial u_r/\partial r)]p' + \sigma_{rr}^a \quad \text{at } r = a \quad (6a)$$

$$f_\theta = -(\partial u_\theta/\partial r)p' + \sigma_{r\theta}^a \quad \text{at } r = a \quad (6b)$$

As discussed in Sec. 2, ΔF_θ, ΔF_r are defined as the circumferential and radial components of the change caused by deformation of the shell surface tractions, taken per unit undeformed surface area. Thus, Eqs. (6), together with the values of the changes caused by deformation for a hydrostatic pressure p recorded in Ref. 2, give

$$\Delta F_\theta = -\frac{p}{a}\left(v - \frac{\partial w}{\partial \theta}\right) + \frac{\partial u_\theta}{\partial r}p' - \sigma_{r\theta}^a \quad \text{at } r = a \quad (7a)$$

$$\Delta F_r = -\frac{p}{a}\left(\frac{\partial v}{\partial \theta} + w - \frac{h}{2a}\frac{\partial^2 w}{\partial \theta^2}\right) + \frac{\partial u_r}{\partial r}p' - \sigma_{rr}^a \quad \text{at } r = a \quad (7b)$$

As will be shown, the changes of the initial shell surface tractions from the core can be related to the middle surface shell displacements, and the shell equilibrium equations (1) can then be used to formulate the buckling condition.

4. Shell Surface Tractions at Shell-Core Interface

It has been shown in the previous section that, for the case of an elastic body subject to an initial state of isotropic, homogeneous stress, the linear theory of elasticity can be employed to determine the additional stresses and displacements. The shell is assumed to be bonded to the elastic core, and the middle surface shell displacements are taken as

$$v = V \sin n\theta \quad (8a)$$

$$w = W \cos n\theta \quad (8b)$$

Then the boundary conditions for the medium are

$$u_\theta = V \sin n\theta - (nh/2a)W \sin n\theta \text{ at } r = a \quad (9a)$$

$$u_r = W \cos n\theta \quad \text{at } r = a \quad (9b)$$

where the second term in Eq. (9a) represents the tangential displacement at the inner surface of the shell caused by the rotation of the shell elements.

Values of the additional stresses and displacements now are calculated from the field equations of classical elasticity for plane strain and the boundary conditions given by Eqs. (9). Following the procedure outlined in Ref. 10, the stress function

$$\phi = (C_1 r^n + C_2 r^{n+2}) \cos n\theta \qquad n \geq 2 \quad (10)$$

where C_1, C_2 are arbitrary constants, yields the following additional stresses and derivatives of the displacements at $r = a$:

$$\sigma_{rr}^a = \frac{E_c \cos n\theta}{a(1 + \nu_c)}\left[HW - G\left(V - \frac{nh}{2a}W\right)\right] \quad (11a)$$

$$\sigma_{r\theta}^a = -\frac{E_c \sin n\theta}{a(1 + \nu_c)}\left[GW - H\left(V - \frac{nh}{2a}W\right)\right] \quad (11b)$$

$$\frac{\partial u_r}{\partial r} = \frac{\cos n\theta}{a}\left[RW - S\left(V - \frac{nh}{2a}W\right)\right] \tag{12a}$$

$$\frac{\partial u_\theta}{\partial r} = \frac{\sin n\theta}{a}\left[LW + M\left(V - \frac{nh}{2a}W\right)\right] \tag{12b}$$

The terms H, G, L, M, R, S in Eqs. (11) and (12) are the following dimensionless quantities:

$$\left.\begin{aligned}
H &= \frac{2n(1-\nu_c) - (1-2\nu_c)}{(1+\nu_c)(3-4\nu_c)} \\
G &= \frac{n(1-2\nu_c) - 2(1-\nu_c)}{(1+\nu_c)(3-4\nu_c)} \\
L &= \frac{n + 4(1-\nu_c)}{3-4\nu_c} \qquad M = \frac{4n(1-\nu_c)+1}{3-4\nu_c} \\
R &= \frac{2n(1-2\nu_c) - 1}{3-4\nu_c} \qquad S = \frac{n - 2(1-2\nu_c)}{3-4\nu_c}
\end{aligned}\right\} \tag{13}$$

Then, Eqs. (7) become

$$\Delta F_\theta = \left\{-\frac{p}{a}\left[V + nW\right] + \frac{p'}{a}\left[LW + M\left(V - \frac{nh}{2a}W\right)\right] + \frac{E_c}{a}\left[GW - H\left(V - \frac{nh}{2a}W\right)\right]\right\}\sin n\theta \tag{14a}$$

$$\Delta F_r = \left\{-\frac{p}{a}\left[nV + \left(1 + \frac{n^2h}{2a}\right)W\right] + \frac{p'}{a}\left[RW - S\left(V - \frac{nh}{2a}W\right)\right] - \frac{E_c}{a}\left[HW - G\left(V - \frac{nh}{2a}W\right)\right]\right\}\cos n\theta \tag{14b}$$

$$\Delta m_\theta = -\frac{h}{2}\left\{\frac{p}{a}\left[V + nW\right] + \frac{p'}{a}\left[LW + M\left(V - \frac{nh}{2a}W\right)\right] + \frac{E_c}{a}\left[GW - H\left(V - \frac{nh}{2a}W\right)\right]\right\}\sin n\theta \tag{14c}$$

Equations (14) give the changes in the surface tractions, caused by deformation in terms of the shell displacements, and the material constants of the medium.

5. Stability Condition

A formula for the critical pressure p now can be obtained from the shell equilibrium equations (1) and the expressions for ΔF_θ, ΔF_r, Δm_θ given by Eqs. (14). Substitution of Eqs. (8) and (14) into Eqs. (1) yields

$$\begin{aligned}
&[(n^2+1)N + n^2E_p + pa - p'aM + aE_cH]V \\
&[+ nE_p + 2n\,N + pan - ap'\,L - aE_cG \times \\
&\qquad (nh/2a)(p'aM - aE_cH)]W = 0
\end{aligned} \tag{15a}$$

$$\begin{aligned}
&[nE_p + 2n\,N + pan + p'aS - E_caG + (nh/2a)(pan + \\
&p'aM - E_c\,Ha)]V\ \{+E_p + (n^2+1)N + (D/a^2)(n^2 - \\
&1)^2 + pa[1 + (n^2h/2a)] - p'a\,R + E_c\,aH + \\
&(nh/2a)[pan + p'a(L-S) - p'a\,M(nh/2a) + \\
&\qquad 2E_c\,a\,G + E_c\,aH(nh/2a)]\}\ W = 0
\end{aligned} \tag{15b}$$

The circumferential stress resultant N is given by

$$N = -(p - p')a \tag{16}$$

where p' is the pressure transmitted to the core. By equating the radial displacements of the shell and core at the position of initial equilibrium, p' may be written in terms of geometrical and material properties of the shell and core and the external hydrostatic pressure p. A routine calculation gives

$$p' = p\alpha/(1+\alpha) \tag{17}$$

where

$$\alpha = [1 - \nu^2/(1+\nu_c)(1-2\nu_c)](E_c/E)(a/h) \tag{18}$$

Equations (15) are linear homogeneous expressions, and the determinant of the coefficients of V, W must vanish for a nontrivial solution. Setting this determinant equal to zero, using Eqs. (16) and (17) to eliminate N and p', and combining the parameters defined by Eq. (13) yield

$$\begin{aligned}
&\left[\frac{p}{p_0(1+\alpha)}\right]^2\Bigg(n(n^2-1)(n+2\alpha) + \alpha\left(\frac{nh}{2a}\right)^2 \times \\
&\quad (n^2+1-2\alpha)\left[\frac{4n(1-\nu_c)+1}{3-4\nu_c}\right] - \frac{nh}{2a}\Bigg\{n(2n^2+1) + \\
&\quad 2n\,\alpha(n^2+n-2) - \alpha^2(n+2) + \alpha(1-\alpha) \times \\
&\quad \left[\frac{n+4(1-\nu_c)}{3-4\nu_c}\right] + 4n\alpha^2\left[\frac{4n(1-\nu_c)+1}{3-4\nu_c}\right]\Bigg\}\Bigg) - \\
&\left[\frac{p}{p_0(1+\alpha)}\right]\Bigg(4\left(\frac{a}{h}\right)^2\Bigg[n^2(n^2-1) + \alpha(n^2-2n-1) + \\
&\quad \frac{4n\,\alpha(1-\nu_c) + 2n^3\,\alpha(1-2\nu_c) - \alpha(n^2-1)}{3-4\nu_c}\Bigg] - \\
&\quad 2n^2\left(\frac{a}{h}\right)\Bigg\{(2n^2-1)(1+\alpha) + 2n\,\alpha - \alpha\left(2 + \frac{n^2h}{2a}\right) \times \\
&\quad \left[\frac{4n(1-\nu_c)+1}{3-4\nu_c}\right]\Bigg\} + \frac{(n^2-1)^2}{3}\Bigg[n^2 - \alpha + \\
&\quad \frac{4n\,\alpha(1-\nu_c)+\alpha}{3-4\nu_c} + 8\,\frac{E_c}{E}(1-\nu^2)\left(\frac{a}{h}\right)^3(n^2-1) \times \\
&\quad \left[\frac{2n(1-\nu_c)+\alpha}{(1+\nu_c)(3-4\nu_c)}\right] + 2n\,\frac{E_c}{E}(1-\nu^2)\left(\frac{a}{h}\right)^2 \times \\
&\left[\left(\frac{nh}{2a}\right)(n^2-1-2\alpha) - 4n\,\alpha\right]\left[\frac{2n(1-\nu_c)-(1-2\nu_c)}{(1+\nu_c)(3-4\nu_c)}\right] + \\
&\quad 2n\,\frac{E_c}{E}(1-\nu^2)\left(\frac{a}{h}\right)^2(2n^2-3\alpha-1) \times \\
&\left[\frac{n(1-2\nu_c)-2(1-\nu_c)}{(1+\nu_c)(3-4\nu_c)}\right]\Bigg) + \Bigg(\frac{4}{3}\left(\frac{a}{h}\right)^2 n^2(n^2-1)^2 + \\
&16\,\frac{E_c}{E}(1-\nu^2)\left(\frac{a}{h}\right)^5(n^2-1)\left[\frac{2n(1-\nu_c)+(1-2\nu_c)}{(1+\nu_c)(3-4\nu_c)}\right] + \\
&\frac{4}{3}(n^2-1)^2(1-\nu)^2\,\frac{E_c}{E}\left(\frac{a}{h}\right)^3\left[\frac{2n(1-\nu_c)-(1-2\nu_c)}{(1+\nu_c)(3-4\nu_c)}\right] + \\
&\quad \left[4\,\frac{E_c}{E}(1-\nu^2)\left(\frac{a}{h}\right)^2\right]^2\left[\frac{n^2-1}{(1+\nu_c)(3-4\nu_c)}\right] + \\
&\quad 8n^2\,\frac{E_c}{E}(1-\nu^2)\left(\frac{a}{h}\right)^4\Bigg\{\frac{n^2h}{2a}\left[\frac{2n(1-\nu_c)-(1-2\nu_c)}{(1+\nu_c)(3-4\nu_c)}\right] + \\
&\quad \frac{2(n^2-1)(1-2\nu_c)}{(1+\nu_c)(3-4\nu_c)}\Bigg\}\Bigg) = 0 \qquad n \geq 2
\end{aligned} \tag{19}$$

where p_0 is the buckling pressure of the shell without an elastic core and is given by

$$p_0 = [E/4(1-\nu^2)](h/a)^3$$

Substitution of a practical range of values into the quadratic equation (19) indicates that one root is much smaller than the other. Thus, the smaller root may be obtained with sufficient accuracy by neglecting the quadratic term in Eq. (19). A further simplification is achieved by noting that, throughout a practical range of parameters, several terms are so small as to be negligible in applications. A simple and accurate expression for the critical pressure of the shell then is given by

$$\frac{p}{p_0(1+\alpha)} = \frac{\dfrac{n^2}{3}(n^2-1) + 4\dfrac{E_c}{E}(1-\nu^2)\left(\dfrac{a}{h}\right)^3\left[\dfrac{2n(1-\nu_c)+(1-2\nu_c)(1+n^2h/a)}{(1+\nu_c)(3-4\nu_c)}\right]}{n^2+\alpha+[2n\,\alpha(1-2\nu_c)-\alpha]/(3-4\nu_c)} \qquad n \geq 2 \qquad (20)$$

The critical pressure is seen to depend on the mode number n, and for each application, the mode number associated with the minimum value of p/p_0 must be determined.

6. Approximate Formulation

In Sec. 3, the changes caused by deformation of the shell surface tractions exerted by the core were calculated from the linearized theory of elasticity. It was shown that changes of the initial surface tractions exerted by the core depend on Poisson's ratio for the core and the mode number, e.g., see Eqs. (13) and (14). An approximate formulation consists in specifying that the initial surface tractions from the core act as a hydrostatic pressure and in neglecting shear stresses at the shell-core interface and the Δm_θ term in Eq. (1b). Thus

$$\Delta F_\theta = -[(p-p')/a][v-(\partial w/\partial\theta)] \qquad (21a)$$

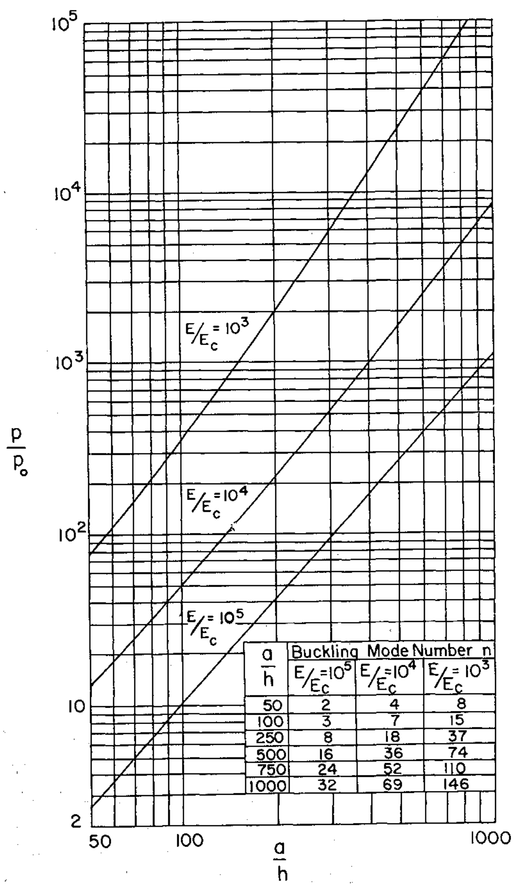
$$\Delta F_r = -[(p-p')/a][(\partial v/\partial\theta)+w] - \sigma_{rr}{}^a \quad \text{at } r=a \qquad (21b)$$

where the changes caused by deformation for a hydrostatic pressure are recorded in Ref. 2.

The value of $\sigma_{rr}{}^a$ may be calculated from the stress function ϕ given by Eq. (10) and the boundary conditions

$$u_r = W\cos n\theta \text{ at } r = a \qquad (22a)$$

$$\sigma_{r\theta}{}^a = 0 \text{ at } r = a \qquad (22b)$$



a/h	Buckling Mode Number n, $E/E_c=10^5$	$E/E_c=10^4$	$E/E_c=10^3$
50	2	4	8
100	3	7	15
250	8	18	37
500	16	36	74
750	24	52	110
1000	32	69	146

Fig. 3. Critical pressure vs radius-to-thickness ratio for $v = 0.3$, $\nu_c = 0.45$.

Following the procedure outlined in Ref. 10, the additional radial stress is

$$\sigma_{rr}{}^a = \frac{E_c(n^2-1)W\cos n\theta}{a(1+\nu_c)[2n(1-\nu_c)-(1-2\nu_c)]} \qquad \text{at } r = a \qquad (23)$$

Substitution of Eqs. (21) and (23) into the shell equilibrium equations (1) and using the assumed shell displacements lead to

$$\frac{p}{p_0(1+\alpha)} = \frac{n^2-1}{3} + \frac{4(1-\nu^2)}{(1+\nu_c)[2n(1-\nu_c)-(1-2\nu_c)]}\left(\frac{E_c}{E}\right)\left(\frac{a}{h}\right)^3 \quad n \geq 2 \qquad (24)$$

Critical pressures predicted by this approximate formula are extremely accurate for high values of Poisson's ratio of the elastic core. A wide range of shell and medium parameters are investigated in the next section for $\nu_c \geq 0.45$, and for all of the cases considered, the differences between the critical pressures and mode numbers predicted by the approximate and exact formulation were negligible. The approximate formula (24) is, however, not accurate for low values of Poisson's ratio.

The excellent accuracy given by the approximate formula (24) for high values of ν_c may be explained by examining Eq. (20). For $\nu_c = 0.5$, the elastic core is incompressible, and the values of α become infinite, but for values of ν_c slightly less than $\frac{1}{2}$, α remains finite. If, in Eq. (20), α is artificially held finite, and ν_c is set equal to $\frac{1}{2}$, Eqs. (20) and (24) are identical.

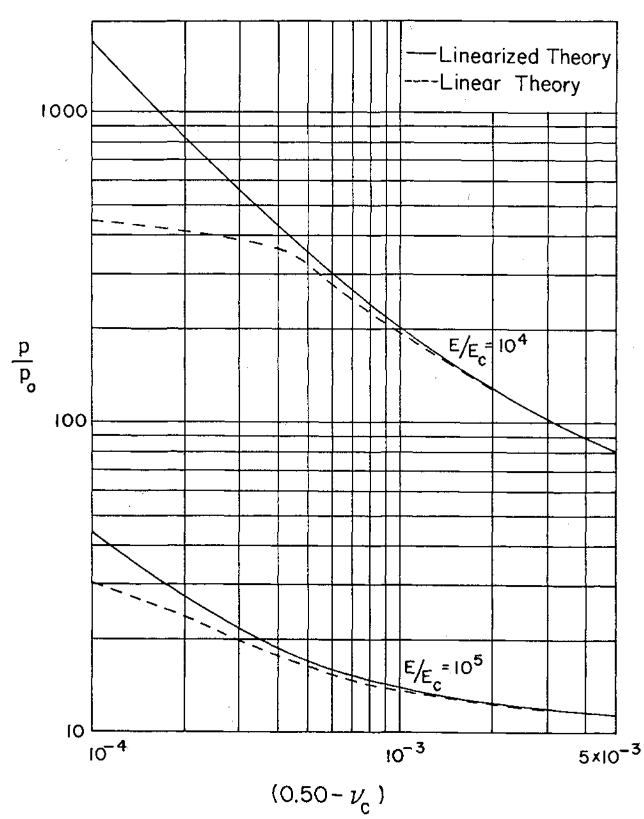


Fig. 4. Critical pressure vs Poisson's ratio for $a/h = 100$, $v = 0.30$.

Thus, for an elastic core with a high value of Poisson's ratio, e.g., $\nu_c \geq 0.45$, shear stresses at the interface are negligible, i.e., the magnitude of $\sigma_{r\theta}{}^a$ is not large enough to influence the buckling pressure of the shell.

7. Numerical Examples

Values of the critical pressure and mode number were calculated for a/h ranging from 50 to 1000, $E/E_c = 10^5$, 10^4, 10^3, and $\nu_c = 0.45$. These results are presented in Fig. 3. It was assumed in the analysis that the shell buckled elastically. To insure that the yield stress of the shell material has not been exceeded, the hoop stress corresponding to the buckling pressure must be calculated. The hoop stress is given by

$$\sigma = [pE/4(1 - \nu^2)p_0(1 + \alpha)](h/a)^2$$

For steel shells and an elastic core with properties $E/E_c = 10^5$, $\nu_c = 0.45$, the hoop stress is $\sigma = 8250$ psi for $a/h = 100$, and $\sigma = 8650$ psi for $a/h = 1000$. If the Young's modulus for the medium is increased to $E/E_m = 10^3$, $\sigma = 188{,}000$ for $a/h = 100$ and $a/h = 1000$.

In Fig. 4, values of the critical pressure for $a/h = 100$ and $E/E_c = 10^5$, 10^4 are given for $0.495 \leq \nu_c \leq 0.4999$. These results are also compared with those obtained by calculating the core reaction with the linear theory, as was done in Ref. 3. No results could be directly compared with those in Ref. 3 because the value of ν_c for a state of plane strain and a solid elastic core were not specified. However, without giving a value for ν_c, the following results are tabulated in Ref. 3:

$$p/p_0 = 52.5 \text{ for } E/E_c = 10^5 \; a/h = 100$$

$$p/p_0 = 485 \text{ for } E/E_c = 10^4 \; a/h = 100$$

Thus, Fig. 4 demonstrates that, for $\nu_c \geq 0.499$, it is necessary to determine the changes caused by deformation of the initial surface tractions exerted by the core.

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